Finite Sections of Periodic Schrödinger Operators

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Discrete Schrödinger Operators

$$H := \Delta_{\mathrm{dis}} + V: \quad \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}), \quad p \in [1, \infty].$$

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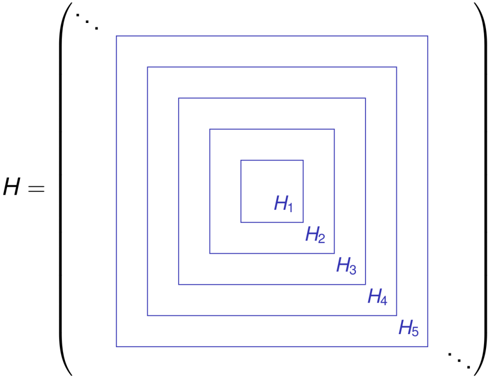
Finite Section Method (FSM) – A Truncation Strategy

Solve $Hx = y$ by solving finite-dimensional matrix-vector equations.



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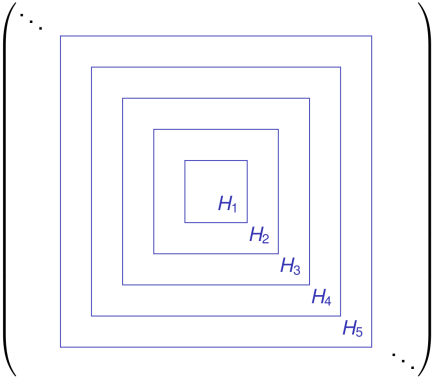

$$\left.\begin{array}{l} H \text{ and } H_m \textit{ invertible} \text{ for all } n \geq n_0\,? \\ H_m^{-1} \to H^{-1} \text{ strongly?} \end{array}\right\} \iff \text{FSM is } \textit{applicable} \text{ to } H$$

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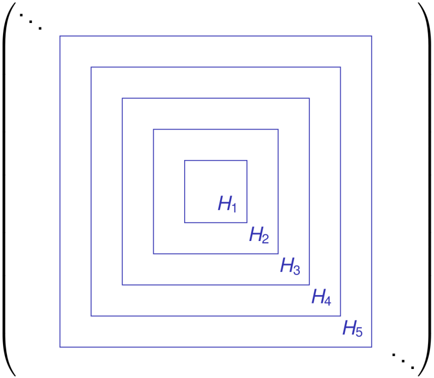

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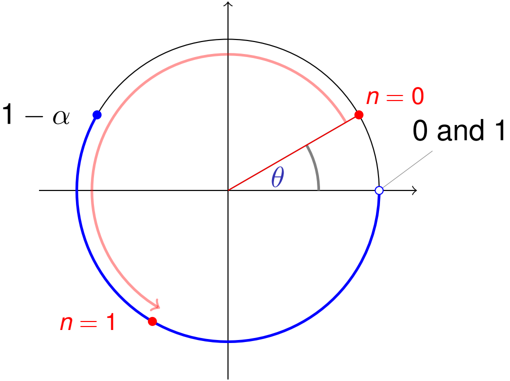

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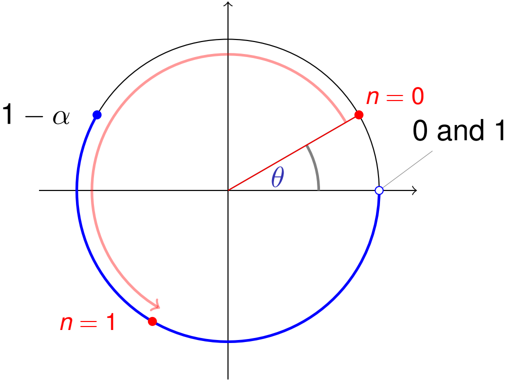

$\alpha \in \mathbb{Q} \implies$ periodic operator
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Continued fraction expansion. Approximate $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by (α_m).

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \quad \text{and} \quad \alpha_m := \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_m}}}.$$

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Spectra.

$$\sigma(H_{\alpha,\theta}) = \bigcap_{k\in\mathbb{N}} \operatorname{clos}\left(\bigcup_{m\geq k} \sigma(H_{\alpha_m,\theta}) \right)$$

Describe Kernel Sequence of H via Transfer Matrices

H is K-periodic Schrödinger Operator.

Let $E \in \mathbb{R}$ and a vector $x \in \ker(H - E)$, then

$$0 = ((H - E)x)_n = x_{n+1} + x_{n-1} + (v(n) - E)x_n\,, \quad n \in \mathbb{Z}\,.$$

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$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} \text{ for all } n \in \mathbb{Z}.$$

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Define *transfer matrix* by:

$$T(n, E) := \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix}.$$

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For periodic potential v with period K, define *monodromy matrix*

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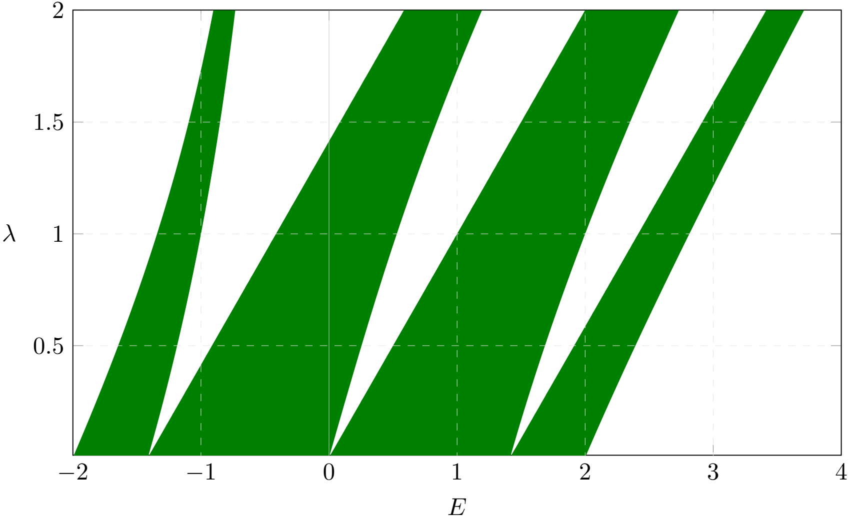
Theorem (Floquet–Bloch Theory)

Let H be a periodic Schrödinger operator and $E \in \mathbb{R}$. Then

$$E \in \sigma(H) \iff |\mathrm{tr}(M(E))| \leq 2\,.$$

Seaweed for the Spectrum of H

$(Hx)_n := x_{n+1} + x_{n-1} + v(n)x_n$, $n \in \mathbb{Z}$, with $v = (\dots, \lambda, \lambda, 0, \lambda, 0, \dots)$ for $\lambda \in \mathbb{R}$.



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H^{R} the Schr.-op. with potential $v^{\mathrm{R}}(n) := v(-n)$, $n \in \mathbb{Z}$. Then

2. $\sigma(B) = \sigma(H)$ *for all* $B \in \operatorname{Lim}(H) \cup \operatorname{Lim}(H^{\mathrm{R}})$.

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Lemma (Chandler-Wilde, Lindner, Seidel, Silbermann, Rabinovich, Roch)

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(i) *The FSM is applicable to H.*

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Theorem (Hagger, 2016)

$$\sigma(H_+) \quad = \quad \sigma(H) \quad \cup \quad \Big\{ E \in \mathbb{R} : m_{21}(E) = 0 \;\; \textit{and} \;\; |m_{11}(E)| < 1 \Big\}$$

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Corollary

$|\operatorname{tr}(M(0))| > 2$. *TFAE:*

(i) *H_+ is invertible on $\ell^p(\mathbb{Z}_+)$.*

(ii) $m_{21}(E) \neq 0 \quad or \quad |m_{11}(E)| > 1$.

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Corollary

$|\operatorname{tr}(M(0))| > 2$. *TFAE:*

(i) H_+ *is* ***not*** *invertible on* $\ell^p(\mathbb{Z}_+)$.

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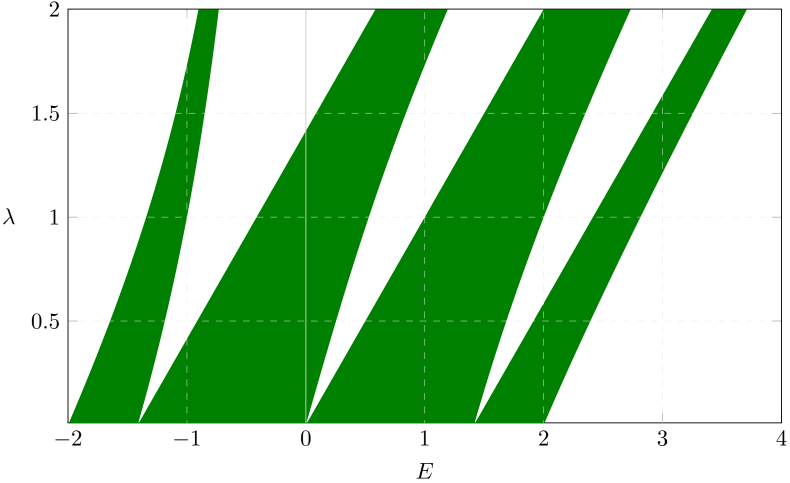
Seaweed for the Spectrum of H_+

$v = (\dots, \lambda, \lambda, 0, \lambda, \dots)$ for $\lambda \in \mathbb{R}$.

$\sigma(H_+) = \sigma(H) \cup \dots$



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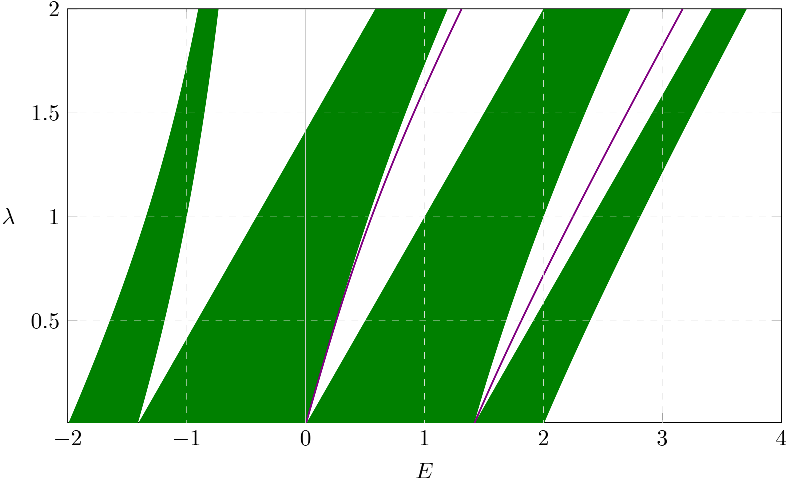

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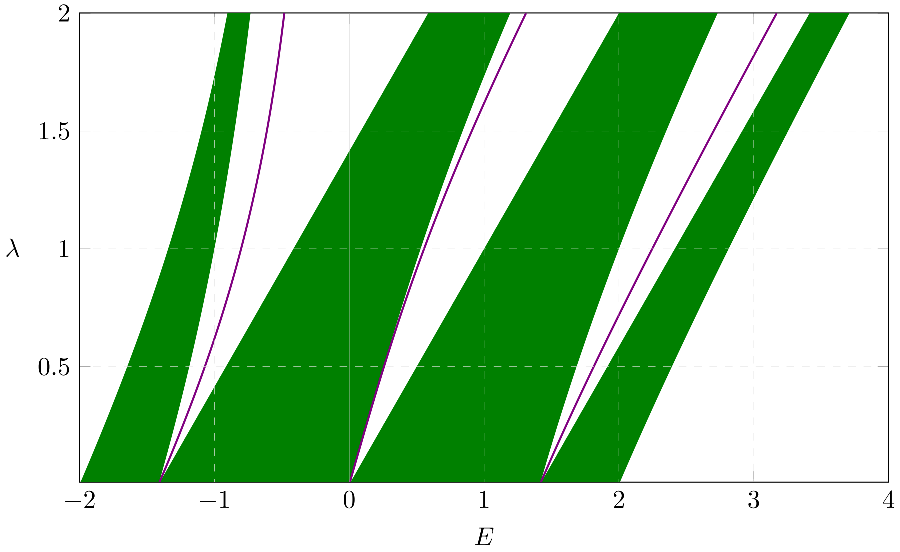
4-periodic Example: No Dirichlet EV at $E = 0$



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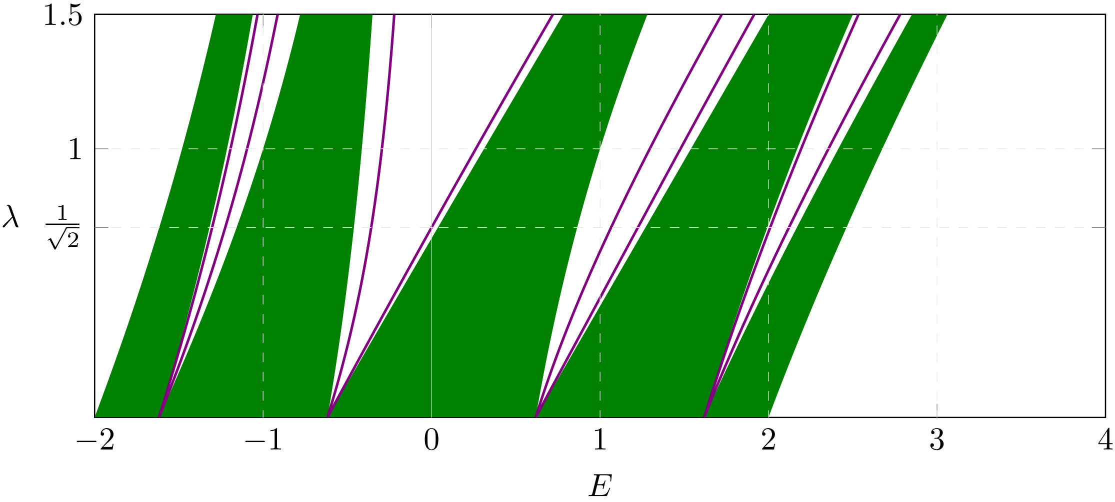

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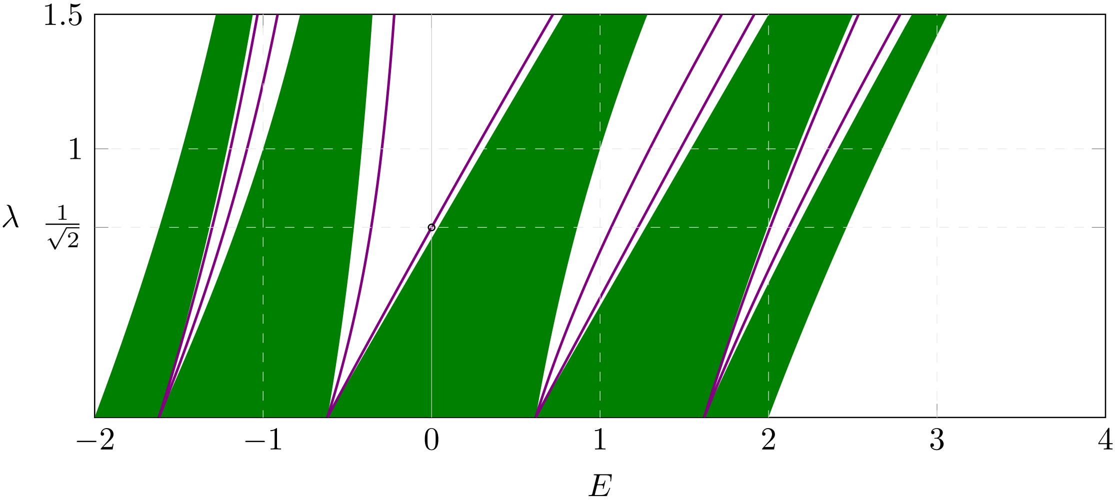

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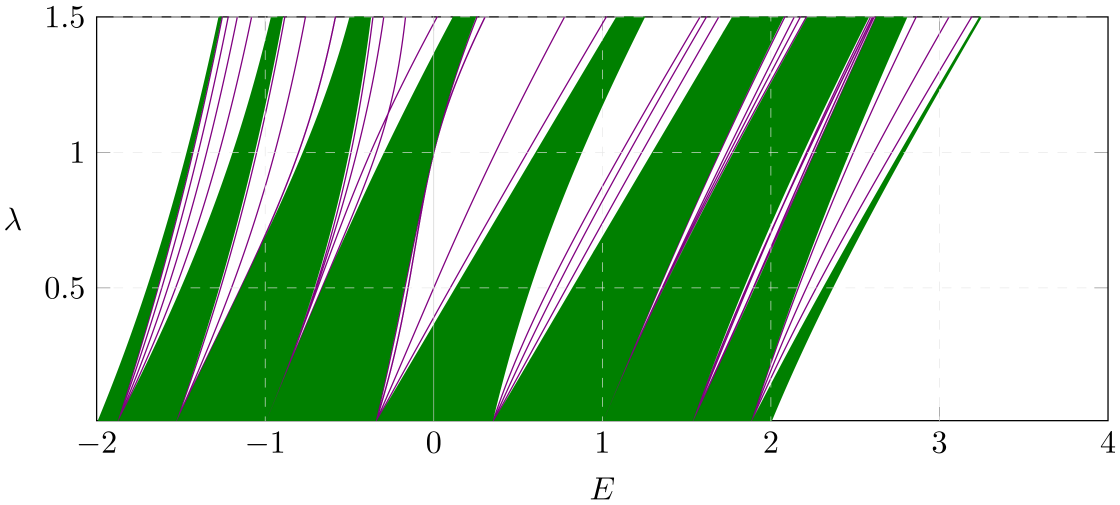

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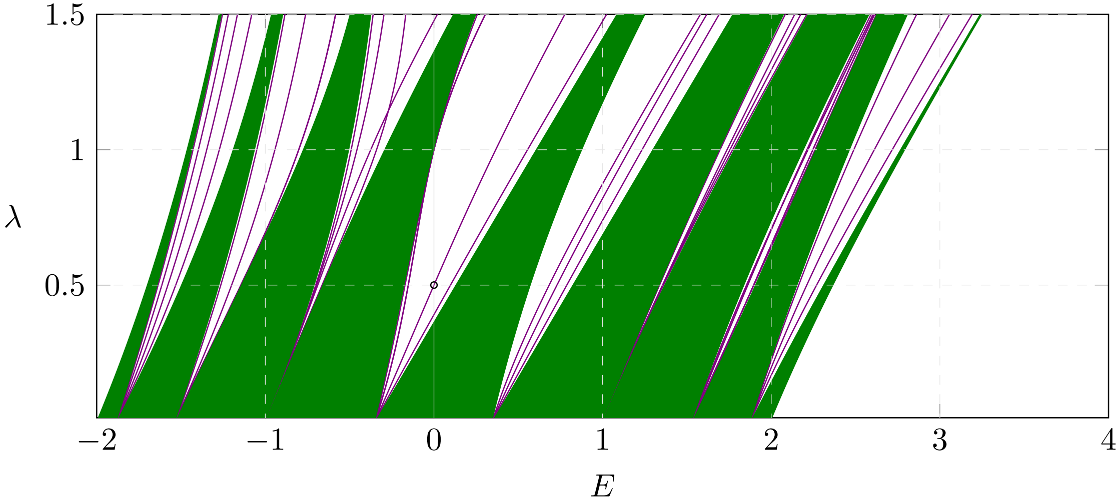

Take Home Messages

Theorem (G., Gallaun, Großmann, Lindner, Ukena)

$p \in [1, \infty]$ *and* $H : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ *a periodic Schrödinger operator. If*

(a) $K \leq 4$, $v(n) \in \{0, \lambda\}$ *for all* $n \in \mathbb{Z}$ *and fixed* $\lambda \in \mathbb{R}$, *or*

(b) $K \leq 8$, $v(n) \in \{0, \lambda\}$ *for all* $n \in \mathbb{Z}$ *and fixed* $\lambda \in \mathbb{Q}$,

then *FSM is applicable to* $H \iff |\mathrm{tr}(M(0))| > 2$.

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- Trace of M can characterize applicability of the FSM.
- Analysis of approximation methods motivates one-sided Schrödinger operators.

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Theorem (G., Gallaun, Großmann, Lindner, Ukena)

$p \in [1, \infty]$ and $H : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ a periodic Schrödinger operator. If

(a) *$K \leq 4$, $v(n) \in \{0, \lambda\}$ for all $n \in \mathbb{Z}$ and fixed $\lambda \in \mathbb{R}$, or*
(b) *$K \leq 8$, $v(n) \in \{0, \lambda\}$ for all $n \in \mathbb{Z}$ and fixed $\lambda \in \mathbb{Q}$,*

then $\quad$ *FSM is applicable to H* $\iff |\operatorname{tr}(M(0))| > 2$.

- Trace of *M* can characterize applicability of the FSM.
- Analysis of approximation methods motivates one-sided Schrödinger operators.
- Invertibility of one-sided Schrödinger ops. $\sim$ avoid zeros of polynomials.

References

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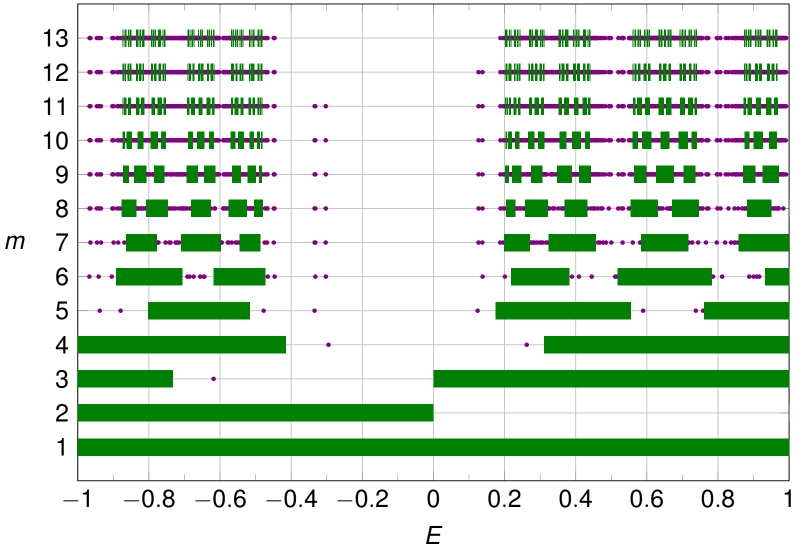
Example: Fibonacci Hamiltonian



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